

# EFFECTIVE DEPHASING FOR A QUBIT INTERACTING WITH A TRANSVERSE CLASSICAL FIELD

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We address the dynamics of a qubit interacting with a quasi static random classical field having both a longitudinal and a transverse component and described by a Gaussian stochastic process. In particular, we analyze in details the conditions under which the dynamics may be effectively approximated by a unitary operation or a pure dephasing without relaxation.

## I. INTRODUCTION

Studying the interaction of a quantum system with its environment plays a fundamental role in the development of quantum technologies. In fact, the quantum features of a system, such as the presence of quantum correlations or superposition of states, are very fragile and may be destroyed by the action of the environmental noise. Decoherence may be induced by classical or quantum noise, i.e. by the interaction with an environment described classically or quantum-mechanically. The classical description is often more realistic to describes environments with a very large number of degrees of freedom, or to describe quantum systems coupled to a classical fluctuating field. Recently, it has also been shown that even certain quantum environments may be described with equivalent classical models [1–3]. Since the environment surrounding a quantum system is often composed by a large number of fluctuators, it is legitimate to assume a Gaussian statistics for the noise [4]. Moreover, the Gaussian approximation is valid even in the presence of non-Gaussian noise, as far as the coupling with the environment is weak [5, 6].

Among the different classes of open quantum systems, a large attention has been paid to qubit systems subject to environmental noise inducing a dephasing dynamics [7–12]. In this framework, in studying the interaction of a qubit with an external field, it is often assumed that the typical frequencies of the system are larger than the characteristic frequencies of the environment. In these situations it is likely that the interaction with the environment induces *decoherence* through dephasing rather than *relaxation* via

damping, i.e. by inducing transitions between the energy levels of the qubit. The effective Hamiltonian describing these kind of processes may be thus written as

$$H(t) = \omega_0 \sigma_z + B_z(t) \sigma_z \quad (1)$$

where,  $\omega_0$  is the natural frequency of the qubit and  $B_z(t)$  is a classical stochastic field with a noise spectrum containing frequencies that are smaller than  $\omega_0$ . The overall evolution of the system is obtained by averaging the unitary evolution governed by the Hamiltonian (1) over the realizations of the stochastic process. The resulting map  $\rho(t) = \mathcal{E}_t(\rho_0)$  corresponds to a pure dephasing which, in turn, leads to a number of interesting phenomena [12], including the abrupt vanishing of entanglement (the so-called entanglement sudden-death [13–15]) and the sudden transition between classical and quantum decoherence [16, 17]. Pure dephasing has been also used to describe the dynamics of qubit systems in colored environments [18, 19] and to quantify their non-Markovian character [20].

In this paper we do not assume the Hamiltonian in Eq. (1), and address the dynamics of a qubit interacting with a Gaussian field with both a longitudinal and a transverse component and with a broad spectrum, possibly including the natural frequency  $\omega_0$  of the qubit. In particular, we are interested in analyzing the conditions under which the dynamics may be effectively approximated by a unitary operation or a pure dephasing without relaxation. Addressing the problem for a generic transverse stochastic field is a challenging task [21, 22] since a high order cumulant expansion is involved. We thus restrict attention to the quasi static regime, where the dynamics of the external field is assumed to be slow, and discuss in some details the conditions to obtain an effective dephasing in this regime.

The paper is structured as follows: In Section II we describe the dynamics of a qubit interacting with an external random classical field having nonzero longitudinal and transverse components. In Section II A we assume a pure transverse field and analyze the conditions under which its effects may be neglected, i.e. the dynamics may be effectively approximated by a unitary operation or a dephasing, whereas in Section II B we consider both components and again analyze the regimes where the dynamics corresponds to dephasing without relaxation. Section III closes the paper with some concluding remarks.

## II. QUBIT INTERACTING WITH A CLASSICAL RANDOM FIELD

Let us consider a two level system interacting with an external fluctuating field  $\vec{B}$ , having both a longitudinal and a transverse component, denoted by  $B_z(t)$  and  $B_x(t)$  respectively. The system Hamiltonian is given by:

$$H(t) = \omega_0 \sigma_z + B_x(t) \sigma_x + B_z(t) \sigma_z, \quad (2)$$

where  $\omega_0$  is the qubit energy and the  $\sigma_i$  are Pauli matrices. Our purpose is to study under which conditions the dynamics governed by the Hamiltonian (2), and by the average over the stochastic processes  $B_z(t)$  and  $B_x(t)$ , may be described by a dephasing map, such that the added term  $B_x(t)\sigma_x$  does not affect the population of the qubit. The time-dependent coefficients  $B_i(t)$  describe stationary Gaussian stochastic processes with zero mean and covariance  $K(t, t') \equiv K(t - t')$ , in formula

$$\begin{aligned} [B_i(t)]_{B_i} &= 0 \\ [B_i(t)B_i(t')]_{B_i} &= K_i(t - t') \quad i = x, z \end{aligned} \quad (3)$$

where the symbol  $[\cdot]_{B_i}$  denotes the average over the process  $B_i(t)$ . A Gaussian process is a process which can be fully described by its second-order statistics. The characteristic function is given by [23]

$$\left[ \exp \left( i \int_{t_0}^t ds J(s) B_i(s) \right) \right]_{B_i} = \exp \left( -\frac{1}{2} \int_{t_0}^t \int_{t_0}^t ds ds' J(s) K_i(s - s') J(s') \right). \quad (4)$$

Upon assuming  $t_0 = 0$ , the evolution operator is expressed as:

$$\begin{aligned} U(t, \omega_0) &= \exp \left\{ -i \mathcal{T} \int_0^t ds H(s) \right\} \\ &\simeq \exp \left\{ -i [\omega_0 t \sigma_z + \varphi_x(t) \sigma_x + \varphi_z(t) \sigma_z] \right\} \end{aligned} \quad (5)$$

where  $\mathcal{T}$  denotes time ordering operator and we have introduced the noise phases

$$\varphi_i(t) = \int_0^t ds B_i(s).$$

The second equality in Eq. (5) is only approximated and is valid upon truncating the Dyson series at the first order, i.e. assuming that we are in the quasi static regime such that the two-time commutator  $[H(t_1), H(t_2)]$  is negligible. If the external field is exactly static, i.e. it is random but it does not change in time, the phases are given by  $\varphi_i(t) = B_i(s) t$  while in the quasi static regime they encompass the effects of the (slow) dynamics of the external field. Because of the Gaussian nature of the considered process, the average of any functional of the noise phase  $g[\varphi(t)]$  may be written as the the average over the process  $\varphi(t)$  with a Gaussian probability distribution:

$$[g(\varphi_i)]_{B_i} = \frac{1}{\sqrt{2\pi\beta_i(t)}} \int d\varphi_i g(\varphi_i) \exp \left\{ -\frac{\varphi_i^2}{2\beta_i(t)} \right\} \quad (6)$$

where we omitted the explicit dependency of  $\varphi$  on time, and the variance function  $\beta(t)$  is defined as:

$$\beta_i(t) = \int_0^t \int_0^t ds ds' K_i(s - s'). \quad (7)$$

The evolution operator may be decomposed into the Pauli basis,  $U(t, \omega_0) = \frac{1}{2} \sum_{j=0}^4 \text{Tr}[U(t, \omega_0) \sigma_j] \sigma_j$ , with  $\sigma_0$  corresponding to the identity matrix  $\mathbb{I}$ , and can thus be expressed as:

$$U(t, \omega_0) = f_I(t, \omega_0) \mathbb{I} + i f_x(t, \omega_0) \sigma_x + i f_z(t, \omega_0) \sigma_z, \quad (8)$$

where

$$f_I(t, \omega_0) = \cos \left[ \sqrt{\varphi_x^2 + (\varphi_z + \omega_0 t)^2} \right] \quad (9)$$

$$f_x(t, \omega_0) = -\frac{\varphi_x \sin \left[ \sqrt{\varphi_x^2 + (\varphi_z + \omega_0 t)^2} \right]}{\sqrt{\varphi_x^2 + (\varphi_z + \omega_0 t)^2}} \quad (10)$$

$$f_z(t, \omega_0) = -\frac{(\varphi_z + \omega_0 t) \sin \left[ \sqrt{\varphi_x^2 + (\varphi_z + \omega_0 t)^2} \right]}{\sqrt{\varphi_x^2 + (\varphi_z + \omega_0 t)^2}}. \quad (11)$$

The qubit density matrix is then evaluated as the average of the evolved density matrix over the stochastic processes  $\vec{B} = \{B_x, B_z\}$ :

$$\rho(t) = \left[ U(t, \omega_0) \rho_0 U^\dagger(t, \omega_0) \right]_{\vec{B}} \quad (12)$$

where  $\rho_0 = \sum_{j,k=1}^2 \rho_{jk} |j\rangle\langle k|$  is the initial density operator. Since the average of any odd terms in  $\varphi_x$  and  $\varphi_z$  in Eq. (12) vanishes, we have

$$\rho(t) = \left[ f_I^2 \rho_0 + f_x^2 \sigma_x \rho_0 \sigma_x + f_z^2 \sigma_z \rho_0 \sigma_z + i f_I f_z [\sigma_z, \rho_0] \right]_{\vec{B}} \quad (13)$$

where we omitted the dependency of the  $f$  functions on  $t$  and  $\omega_0$ . After performing the average in Eq. (13), the evolved density matrix may be rewritten as:

$$\rho(t) = A_I \rho_0 + A_x \sigma_x \rho_0 \sigma_x + A_z \sigma_z \rho_0 \sigma_z + i A_{Iz} [\sigma_z, \rho_0] \quad (14)$$

where:

$$A_i = A_i(t, \omega_0) = \left[ f_i(t, \omega_0)^2 \right]_{\vec{B}} \quad i = I, x, z \quad (15)$$

$$A_{Iz} = A_{Iz}(t, \omega_0) = \left[ f_I(t, \omega_0) f_z(t, \omega_0) \right]_{\vec{B}} \quad (16)$$

and the condition  $A_I + A_x + A_z = 1$  must be satisfied to preserve unitarity. Upon writing explicitly the density matrix

$$\rho(t) = \begin{pmatrix} (A_I + A_z) \rho_{11} + A_x \rho_{22} & (A_I + 2i A_{Iz} - A_z) \rho_{12} + A_x \rho_{21} \\ A_x \rho_{21} + (A_I - 2i A_{Iz} - A_z) \rho_{21} & A_x \rho_{11} + (A_I + A_z) \rho_{22} \end{pmatrix} \quad (17)$$

one immediately sees that whenever  $A_x$  is vanishing or may be neglected, the Hamiltonian (2) leads to a dephasing map, with a complex dephasing coefficient. In the next Section, we analyze whether this is true also in other conditions.

### A. Interaction with a pure transverse field

In order to gain insight into the dynamics of the system let us first consider the case of zero longitudinal field  $B_z(t) = 0$  and look for the conditions under which the effects of the transverse field may be neglected or subsumed by a dephasing. We set  $\varphi_x = \varphi$  and evaluate  $A_x(t)$  from Eq. (15), which now reads

$$A_x(t, \omega_0, \beta) = \frac{1}{\sqrt{2\pi\beta(t)}} \int_{-\infty}^{\infty} d\varphi \varphi^2 \frac{\sin^2 \left[ \sqrt{\varphi^2 + (\omega_0 t)^2} \right]}{\varphi^2 + (\omega_0 t)^2} \exp \left( -\frac{\varphi^2}{2\beta(t)} \right), \quad (18)$$

where the exact functional form of the variance  $\beta(t)$  depend on the specific features of the process  $B_x(t)$ . Upon inspecting Eq. (18) one sees that  $A_x(t, \omega_0, \beta)$  vanishes whenever  $\omega_0 t \gg 1$  or  $\beta(t) \ll 1$ . The first condition corresponds to the assumption of a large qubit frequency (outside the spectrum of the noise), whereas the second one  $\beta \ll 1$  is related to the specific properties of the stochastic process describing the noise. In order to better understand the effects of the transverse field, we now evaluate the function  $A_x(t, \omega_0, \beta)$  from Eq. (18) for three classical Gaussian processes with Ornstein-Uhlenbeck (OU), Gaussian (G) and power-law (PL) autocorrelation function, i.e.

$$K_{OU}(t - t', \gamma, \Gamma) = \frac{1}{2} \Gamma \gamma e^{-\gamma|t-t'|} \quad (19)$$

$$K_G(t - t', \gamma, \Gamma) = \frac{1}{\sqrt{\pi}} \Gamma \gamma e^{-\gamma^2(t-t')^2} \quad (20)$$

$$K_{PL}(t - t', \gamma, \Gamma, \alpha) = \frac{1}{2} (\alpha - 1) \Gamma \gamma \frac{1}{(\gamma|t - t'| + 1)^\alpha} \quad (21)$$

which, by Eq. (7), give:

$$\beta_{OU}(\tau, R_\Gamma) = R_\Gamma (\tau - 1 + e^{-\tau}) \equiv R_\Gamma g_{OU}(\tau), \quad (22)$$

$$\beta_G(\tau, R_\Gamma) = \frac{R_\Gamma}{\sqrt{\pi}} \left[ e^{-\tau^2} - 1 + \sqrt{\pi} \tau \text{Erf}(\tau) \right] \equiv R_\Gamma g_G(\tau), \quad (23)$$

$$\beta_{PL}(\tau, R_\Gamma, \alpha) = R_\Gamma \frac{(1 - \tau)^2 + (1 + \tau)^\alpha [\tau(\alpha - 2) - 1]}{(1 + \tau)^\alpha (\alpha - 2)} \equiv R_\Gamma g_{PL}(\tau), \quad (24)$$

where  $\Gamma$  and  $\gamma$  are the *damping* and the *memory* parameters of the processes,  $\tau = \gamma t$  denotes the rescaled dimensionless time,  $R_\Gamma = \frac{\Gamma}{\gamma}$ ,  $\alpha > 2$  is a real number and  $\text{Erf}(x)$  is the error function. The (quasi) static limit is obtained for vanishing  $\gamma$  keeping  $\Gamma\gamma$  finite. The  $g_x(\tau)$ 's are functions of the sole rescaled time,  $x = OU, G, PL$ . We have numerically evaluated the integral in Eq. (18) for the three different process as a function of rescaled time  $\tau$  and the two ratios  $R_\omega = \omega_0/\gamma$  and  $R_\Gamma$ . In particular, we want to see when  $A_x(\tau, R_\omega, R_\Gamma)$  is negligible, as a function of the parameters  $R_\omega$  and  $R_\Gamma$ , and to this aim, we have maximized the function over the time  $\tau$  and determined where the maximum is smaller than a given threshold. In Fig 1 we show the region in the  $R_\omega$ - $R_\Gamma$  plane where  $\max_\tau |A_x(\tau, R_\omega, R_\Gamma)| < 10^{-3}$  for the three different

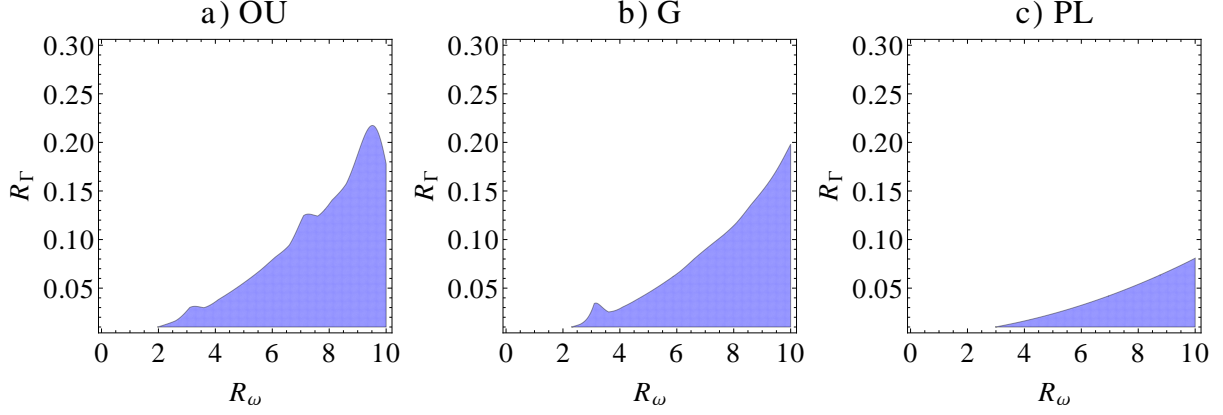


FIG. 1. Region where the coefficient  $\max_{\tau} |A_x(\tau, R_{\omega_0}, R_{\Gamma})| < 10^{-3}$  for three different processes characterized by a) exponential b) Gaussian and c) power law ( $\alpha = 4$ ) autocorrelation function

processes. As it is apparent from the plots, the coefficient is negligible if  $R_{\omega} \gg 1$  and/or  $R_{\Gamma} \ll 1$ , with the specific ranges depending on the chosen process.

In Fig. 1c, we have shown results for a powerlaw process with  $\alpha = 4$ . This is a good representative of the family (21), since different values of the parameter lead to the same conditions for an effective dephasing. The behaviour emerging from Fig. 1 is in agreement with the qualitative considerations made above and with the fact that the condition  $\beta \ll 1$  is equivalent to  $R_{\Gamma} \ll 1$ .

Since we assumed Gaussian processes with zero mean, we can Taylor-expand the function  $f_x(t, \omega_0)^2$  around  $\varphi = 0$ . By dropping the expansion at the second order we may analytically compute the integral (15) and obtain:

$$\tilde{A}_x(t, \omega_0) \simeq \beta(t) \frac{\sin^2 \omega_0 t}{(\omega_0 t)^2}. \quad (25)$$

From Eq. (25) we immediately see that the coefficient  $A_x(t, \omega_0)$  vanishes for vanishing  $\beta(t)$  or for  $R_{\omega} \tau \equiv \omega_0 t \gg 1$ . This is in agreement with the numerical results and shows that a second order expansion is sufficient to capture the two regimes where the effects of the transverse field on the populations may be neglected. In order to gain more insight on the possible differences between the two regimes we expand, up to second order in  $\varphi$ , also the other  $f$  functions, arriving at

$$\tilde{A}_I(t, \omega_0) \simeq \cos^2 \omega_0 t - \beta(t) \frac{\sin 2\omega_0 t}{2\omega_0 t} \quad (26)$$

$$\tilde{A}_z(t, \omega_0) \simeq \sin^2 \omega_0 t - \beta(t) \left( \frac{\sin^2 \omega_0 t}{(\omega_0 t)^2} - \frac{\sin 2\omega_0 t}{2\omega_0 t} \right) \quad (27)$$

$$\tilde{A}_{Iz}(t, \omega_0) \simeq -\frac{1}{2} \sin 2\omega_0 t - \beta(t) \left( \frac{\cos 2\omega_0 t}{2\omega_0 t} - \frac{\sin 2\omega_0 t}{4(\omega_0 t)^2} \right). \quad (28)$$

In turn, the coefficient in the off-diagonal elements of the density matrix reads as follows

$$A_I + 2i A_{Iz} - A_z \simeq e^{-2i\omega_0 t} + \frac{\beta(t)}{2(\omega_0 t)^2} \quad R_\omega \gg 1 \quad (29)$$

$$\simeq e^{-2i\omega_0 t} \left[ 1 - \frac{\beta(t)}{2(\omega_0 t)^2} - i \frac{\beta(t)}{\omega_0 t} \right] + \frac{\beta(t)}{2(\omega_0 t)^2} \quad R_\Gamma \ll 1. \quad (30)$$

The above expressions, together with Eq. (25) which is valid in both the limiting cases, illustrate the qualitative differences between the two regimes: for  $R_\omega \gg 1$  the leading terms in Eqs. (29) and (25) are the same, meaning that either relaxation occurs or the dynamics is unitary, whereas for  $R_\Gamma \ll 1$  the multiplicative term in Eq. (30) reveals that the effective dynamics of the qubit corresponds to a dephasing. The expressions above correspond to situations where the effective dynamics is valid at all times. More generally, it may happen that the weaker conditions  $R_\omega \tau \gg 1$  and  $R_\Gamma g_x(\tau) \ll 1$  are satisfied for up to some values of  $\tau$ , corresponding to regimes where the effective dynamics appears only for a finite interaction time.

### B. Effective dephasing in the general case

We now consider the complete Hamiltonian (2), with the longitudinal term  $B_z(t) \neq 0$ . The coefficient  $A_x$ , in this case, takes the form:

$$A_x(\beta_x, \beta_z, t, \omega_0) = \frac{1}{2\pi \sqrt{\beta_x(t)\beta_z(t)}} \iint d\varphi_x d\varphi_z \exp \left( -\frac{\varphi_x^2}{2\beta_x(t)} - \frac{\varphi_z^2}{2\beta_z(t)} \right) f_x^2(\varphi_x, \varphi_z, t, \omega_0) \quad (31)$$

where we explicitly wrote the dependency on the  $\beta$  functions. Following the line of reasoning of the previous section, we expand the function  $f_x$  in Eq. (10) around  $\varphi_x = 0$  and  $\varphi_z = 0$ , and we drop the expansion at the second order. Inserting this expansion in Eq. (15), we are able to write the analytical expression for  $\tilde{A}_x$ :

$$\tilde{A}_x(\beta_x, \beta_z, t, \omega_0) = \beta_x(t) \frac{\sin^2(2\omega_0 t)}{\omega_0^2 t^2} + \frac{\beta_x(t)\beta_z(t)}{2(\omega_0 t)^4} \left[ 3 + (2\omega_0^2 t^2 - 3) \cos 2\omega_0 t - 4\omega_0 t \sin 2\omega_0 t \right] \quad (32)$$

Upon expanding to the second order all the terms we may write the analytical expression of the evolved density matrix, where the off-diagonal coefficient  $K = A_I + 2i A_{Iz} - A_z$  is given by:

$$K \simeq e^{-2i\omega_0 t} [1 - 2\beta_z(t)] + \frac{\beta_x(t)}{2\omega_0^2 t^2} \quad R_\omega \gg 1 \quad (33)$$

$$\simeq e^{-2i\omega_0 t} \left[ 1 - 2\beta_z(t) - \frac{\beta_x(t)}{2\omega_0^2 t^2} - i \frac{\beta_x(t)}{\omega_0 t} \right] + \frac{\beta_x(t)}{2\omega_0^2 t^2} \quad R_\Gamma \ll 1. \quad (34)$$

Looking at Eq. (33) and Eq. (34), one sees that when  $R_\omega \gg 1$  one may just neglect the effects of the transverse field, whereas for  $R_\Gamma \ll 1$  one has an additional effective term in the coefficient  $K$ . As for the previous case, the effective dynamics emerges if the above conditions are valid at all times. More general regimes can be written as  $R_\omega \tau \gg 1$  and  $R_\Gamma g_x(\tau) \ll 1$ .

### III. CONCLUSIONS

The effect of classical noise on a qubit system may be described as the interaction with a random field. In this paper we analyzed, in the quasi static regime, the conditions under which a general dynamics, including interaction with a transverse field, may be approximated by an effective dephasing, without changes in the populations. In particular, we studied the time evolution of a qubit subject to a transverse and longitudinal field. We found that the properties of the stochastic processes analyzed, i.e. the autocorrelation function, play a role, through the variance function  $\beta(t)$ . Whenever this function is small, the dynamics can be described as a dephasing. Moreover, we recovered the known condition of large system's energy,  $\omega_0 t \gg 1$ , which prevents jumps between the qubit levels. If these assumptions do not hold, the general dynamics is not a dephasing and relaxation phenomena may occur, with changes in the qubit populations as described by Eq. (17).

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